2 Distributions

Exercise 2.1. Let $E : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be the function defined by

$$E(x) = \frac{1}{2\pi} \log |x|.$$

Show that

i) $E \in \mathscr{D}'(\mathbb{R}^2)$;

ii) for every $\varepsilon > 0$ one has that $\operatorname{div} \nabla E = 0$ on $\mathbb{R}^2 \setminus \overline{B}(0, \varepsilon)$.

Exercise 2.2. Consider the function E introduced in Exercise 2.1. Show that for every $\varphi \in \mathscr{D}(\mathbb{R}^2)$ it holds

$$\int_{\mathbb{R}^2} E(x) \Delta\varphi(x)\, dx = \varphi(0)$$

and deduce that E is a fundamental solution for the Laplacian operator Δ on $\mathscr{D}'(\mathbb{R}^2)$, namely show that $\Delta E = \delta_0$ in the sense of distributions.

Exercise 2.3. Provide an example of a distribution $T \in \mathscr{D}'(\mathbb{R})$ such that

$$T(\varphi) = \int_0^\infty \frac{\varphi(x)}{x} dx \quad \forall \varphi \in \mathscr{D}((0,\infty)) \qquad \text{and} \qquad T(\varphi) = 0 \quad \forall \varphi \in \mathscr{D}((-\infty, 0)).$$

Exercise 2.4. Let $\varepsilon_n^+, \varepsilon_n^-$ be two sequences of positive numbers converging to 0. Assume that there exists $a \in (0, \infty)$ such that $\frac{\varepsilon_n^+}{\varepsilon_n^-} \underset{n\to\infty}{\longrightarrow} a$. Show that the sequence of distributions

$$T_n(\varphi) = \int_{-\infty}^{-\varepsilon_n^-} \frac{\varphi(x)}{x} dx + \int_{\varepsilon_n^+}^{+\infty} \frac{\varphi(x)}{x} dx$$

converges and compute the limit distribution.

Exercise 2.5. Show that the function $g(x) = \max(0, x)$ is a fundamental solution for the one-dimensional differential operator $L = \dfrac{\partial^2}{\partial x^2}$.